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NOTE

EDGE-COLOURINGS OF $K_{n,n}$ WITH NO LONG TWO-COLOURED CYCLES

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Consider the set of all proper edge-colourings of a graph G with n colours. Among all such colourings, the minimum length of a longest two-coloured cycle is denoted $L(n,G)$. The problem of understanding $L(n,G)$ was posed by Häggkvist in 1978 and, specifically, $L(n,K_{n,n})$ has received recent attention. Here we construct, for each prime power $q \geq 8$, an edge-colouring of $K_{n,n}$ with n colours having all two-coloured cycles of length $\leq 2q^2$, for integers n in a set of density $1-3/(q-1)$. One consequence is that $L(n,K_{n,n})$ is bounded above by a polylogarithmic function of n, whereas the best known general upper bound was previously $2n-4$.

1. Introduction

Given a proper edge-colouring γ of a graph G with n colours, let $L_\gamma(n,G)$ denote the length of a longest cycle which is two-coloured by γ. Define $L(n,G) = \min L_\gamma(n,G)$, where the minimum is taken over all proper edge-colourings γ with n colours. For $m \geq n$, observe that $L(m,G) \leq L(n,G)$, provided these quantities make sense. This follows since any proper colouring with n colours achieving $L(n,G)$ can be modified by "splitting colour classes" to use additional colours without lengthening any bicoloured cycle. So, from the viewpoint of upper bounds on $L(n,G)$, it is natural to take n as the chromatic index of G.

Since determining $L(n,G)$ appears difficult in general, specific families of graphs attract the most attention. Here, we consider the family of complete

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bipartite graphs $K_{n,n}$. The problem of determining $L(n,K_{n,n})$ was raised by Häggkvist in [2]. These numbers, and more generally $L(n,G)$, have become fairly well-known as *Häggkvist numbers*.

Based on work of Cameron [1], it follows that $L(n,K_{n,n})=4$ if and only if $n=2^k$, $k\geq 1$. It was recently shown by Ninčák and Owens in [4] that $L(n,K_{n,n})\leq 2p$ when $n=p^k$. They also proved the general bound $L(n,K_{n,n})\leq 2n-4$, for all $n\notin\{2,3,5\}$. Here, using linear spaces, we provide a construction of edge-colourings of $K_{n,n}$ which substantially improves this upper bound. This is found in Theorem 4 below, and consequences are discussed in the conclusion.

On the other hand, lower bounds on $L(n,K_{n,n})$ appear much more difficult. The best known general lower bound is simply $L(n,K_{n,n})\geq 6$ when $n\neq 2^k$, again due to Cameron's result. We do not pursue lower bounds here, except for a brief discussion and conjecture at the conclusion of this note.

2. Linear Spaces and Good Colourings

A *linear space* is a pair $(X,\mathcal{L})$, where X is a set of *points* and $\mathcal{L}$ is a set of *lines*, or subsets of X, with the property that every line contains at least two points and any two distinct points are contained in exactly one line. Linear spaces are also known as *pairwise balanced designs*.

We now recall an important family of linear spaces. Let q be a prime power and $\mathbb{F}_q$ the finite field of order q. Consider the vector space $V=\mathbb{F}_q^{d+1}$ and define $X=\{U_1,\dots,U_{1+q+\cdots+q^d}\}$ as the set of all subspaces of dimension 1. For each subspace $W\subseteq V$ denote by B_W the set of all $U_j\in X$ with $U_j\subseteq W$. Let $\mathcal{L}$ be the set of all B_W, where W has dimension 2. Then $(X,\mathcal{L})$ is a linear space with $1+q+\cdots+q^d$ points and with every line of size $q+1$. For $d=2$, $(X,\mathcal{L})$ has the additional property that every pair of lines intersects in exactly one point. This linear space (or more generally the family of all B_W, where $\dim(W)\geq 2$) forms the *projective space of dimension d* over $\mathbb{F}_q$, or $\mathrm{PG}_d(q)$.

Given a linear space $(X,\mathcal{L})$, a subset $Y\subseteq X$ induces a linear space $(Y,\mathcal{L}')$, called a *truncation* of $(X,\mathcal{L})$, where $\mathcal{L}'$ consists of all $L\cap Y$, where $L\in\mathcal{L}$ and $|L\cap Y|\geq 2$. Let $W\subset\mathbb{F}_q^{d+1}$ be a subspace of codimension 1, and consider $Y=\{U_j:U_j\not\subseteq W\}\subset X$ in $\mathrm{PG}_d(q)$. The truncation of $\mathrm{PG}_d(q)$ with respect to Y is the *affine space of dimension d* over $\mathbb{F}_q$, or $\mathrm{AG}_d(q)$. Every line in $\mathrm{AG}_d(q)$ is on q points, and the set of lines can be partitioned into $1+q+\cdots+q^{d-1}$ classes, each of which consists of q^{d-1} parallel lines. For convenience, we may represent the points of $\mathrm{AG}_q(d)$ as elements of the vector space $\mathbb{F}_q^d$, with lines determined by affine subsets of dimension 1. The following results concern the existence of truncations of $\mathrm{AG}_d(q)$ and $\mathrm{PG}_d(q)$, leaving no lines of size two.

Lemma 1. *Let $q \geq 5$ be a prime power and $d \geq 2$ an integer. If there exist truncations of $\mathrm{AG}_{d-1}(q)$ on each of m and m' points and with no lines of size two, then there exists a truncation of $\mathrm{AG}_d(q)$ on $q^{d-1}s+m+m'$ points for each $s=3,4,\ldots,q-2$, and with no lines of size two.*

Proof. Let $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ be truncations of $\mathrm{AG}_{d-1}(q)$ on m, m' points, respectively, with no lines of size two. List the elements of $\mathbb{F}_q$ as $e_1,\ldots,e_q$. Consider the points $X=\mathbb{F}_q^{d-1}\times\{e_1,\ldots,e_s\}\cup Y\times\{e_{s+1}\}\cup Y'\times\{e_{s+2}\}$, regarded as a subset of $\mathbb{F}_q^d$. We have $|X|=q^{d-1}s+m+m'$. The projection onto the dth coordinate of any line in $\mathrm{AG}_d(q)$ is either $\{e_1,\ldots,e_q\}$ or $\{e_i\}$ for $1\leq i\leq q$. In the first case, such lines intersect at least s points $e_1,\ldots,e_s$ of X. In the latter case, such lines are contained in X for $i\leq s$, disjoint from X for $i>s+2$, and intersect at least 3 points of $Y\times\{e_{s+1}\}$ or $Y'\times\{e_{s+2}\}$ by hypothesis. ∎

Theorem 2. *Let $q\geq 8$ be a prime power and $d\geq 1$ an integer. For every integer n, $4q^{d-1}\leq n\leq q^d$, there exists a truncation of $\mathrm{AG}_d(q)$ on n points, and with no lines of size two.*

Proof. We proceed by induction on d. For $d=1$, the result is clear, since $\mathrm{AG}_1(q)$ is a single line of size q, which has truncations on each of $4,\ldots,q$ points. Let $\delta\geq 2$ and suppose the result is true for $d=\delta-1$. Since $q\geq 8$, any integer n with $4q^{\delta-1}\leq n\leq q^\delta$ can be written $n=q^{\delta-1}s+m+m'$, where $s\geq 3$ and $4q^{\delta-2}\leq m,m'\leq q^{\delta-1}$. By Lemma 1, there exists a truncation of $\mathrm{AG}_\delta(q)$ on n points. ∎

Fix two distinct points x and y of a linear space $(X,\mathcal{L})$, and let L be the unique line containing $\{x,y\}$. Define the graph G_{xy} to have vertex set $X\setminus L$ with $\{w,z\}$ an edge whenever either $\{w,z,x\}$ or $\{w,z,y\}$ are collinear in $\mathcal{L}$.

Lemma 3. *Let d be a positive integer. If a linear space $(X,\mathcal{L})$ is obtained by truncating $\mathrm{PG}_d(q)$, then for any $x,y\in X$, every component of G_{xy} has size at most q^2.*

Proof. It suffices to prove the statement in the case when $(X,\mathcal{L})$ is $\mathrm{PG}_d(q)$, since truncation does not increase the sizes of components in G_{xy}. The case $d\leq 2$ is trivial, so suppose $d\geq 3$. Consider $\mathrm{PG}_d(q)$ with point set X. Let $Y\subset X$ be any set of points containing x,y inducing a copy of $\mathrm{PG}_{d-1}(q)$. The q^d vertices in $X\setminus Y$ induce a copy of $\mathrm{AG}_d(q)$. Each line in Y is on a class of q^{d-2} parallel affine planes through $X\setminus Y$. Therefore, all components of the subgraph of G_{xy} induced by $X\setminus Y$ have size at most q^2. By induction, we may assume the subgraph induced by $Y\setminus\{x,y\}$ has the same property, and the result follows. ∎

It is notable that the cycle bound in Lemma 3 is independent of d. We now state and prove our main construction of proper edge-colourings of $K_{n,n}$ using linear spaces.

Theorem 4. *Let $(X,\mathcal{L})$ be a linear space with $|X|=n$ and $3\le|L|\le l$ for any $L\in\mathcal{L}$. Suppose further that the size of the largest component of any G_{xy} is at most k. Then there exists a proper edge-colouring of $K_{n,n}$ with n colours having no two-coloured cycle of length greater than $2\max\{k,l\}$.*

Proof. We construct an edge-colouring of the complete bipartite graph with bipartition $(X\times\{1\},X\times\{2\})$. For each $L\in\mathcal{L}$ with $|L|=m$, there exists a proper edge-colouring γ_L of $K_{m,m}$ with m colours on $(L\times\{1\},L\times\{2\})$ such that the edges $\{(x,1),(x,2)\}$, $x\in L$, each receive a different colour c_x. (Such a colouring is easy to find for $m\neq 2$.) Define $\gamma(\{(x,1),(x,2)\})=c_x$ and for $x\neq y$, $\gamma(\{(x,1),(y,2)\})=\gamma_L(\{(x,1),(y,2)\})$, where L is the unique line on x,y. Observe that γ is well-defined because $(X,\mathcal{L})$ is a linear space. Now suppose G is the graph induced by two distinct colours c_x and c_y. Let L be the line through points x,y. The subgraph of G induced by $L\times\{1,2\}$ is 2-regular, and so has all components of size $\le 2|L|\le 2l$. The subgraph of G induced by $(X\setminus L)\times\{1,2\}$ consists of edges of the form $\{(w,1),(z,2)\}$, where $\{w,z\}\in G_{xy}$. So each component of this subgraph has size at most $2k$. ∎

3. Summary and Conclusion

For a fixed prime power $q\ge 8$, Theorem 2 states that the set of n for which there is a truncation of some $\mathrm{PG}_d(q)$ on n points, and with no line of size two, has density at least

$$(q-4)\frac{1/q}{1-1/q}=1-\frac{3}{q-1}.$$

For such values of n, an application of Theorem 4 yields $L(n,K_{n,n})\le 2\max\{q^2,q+1\}=2q^2$, again independent of d. By merely using prime powers of the form $q=4^f$, we arrive at the following simple bound.

Theorem 5. *Let $\lambda(k)$ be the least common multiple of the first k positive integers. If $n<4^{\lambda(k)}$, then $L(n,K_{n,n})\le 4^{2k+1/2}$.*

Proof. We may assume $n > 4^{2k}$, for otherwise the conclusion is trivial. Say that an integer $f \geq 2$ "fails" for n if f divides $\lfloor \log_4 n \rfloor$. Now if $\log_4 n < \lambda(k)$, then not all of the integers $2, 3, \ldots, k$ fail for n. If f does not fail for n, then $4^{f(d-1)+1} \leq n < 4^{fd}$ for some integer $d \geq 2$. By Theorem 2, some truncation of $\mathrm{PG}_d(4^f)$ has exactly n points and no line of size two. By Lemma 3 and Theorem 4, $L(n, K_{n,n}) \leq 2 \cdot (4^f)^2 \leq 4^{2k+1/2}$. ∎

It is well-known (see [3]) that the *Chebyshev function* $\ln(\lambda(k))$ is asymptotic with k. So an easy consequence of Theorem 5 is the polylogarithmic bound

$$L(n, K_{n,n}) \leq [\log(n)]^{4 \ln 2 + o(1)}.$$

However, it should be mentioned that this is far from best possible. Using prime powers q of the form 4^f and 5^f, we have $L(n, K_{n,n}) \leq 2 \cdot 5^{2k}$ unless $\lambda(k)$ simultaneously divides $\lfloor \log_4 n \rfloor$ and $\lfloor \log_5 n \rfloor$. Lower bounds on n follow from rational approximations of $\log_4 5$, but based on Conjecture 6 below, this method is probably unworthy of further attention.

A more intricate argument improves the interval in Theorem 2, and one obtains

$$L(n, K_{n,n}) \leq 2q^2$$

provided there exists an integer $d \geq 2$ with $3q^{d-1} \leq n \leq 1 + q + \cdots + q^d$. For a concrete example, using prime powers $q \leq 13$ we have calculated $L(n, K_{n,n}) \leq 338$ for all $n < 10^{15}$.

The details of a general argument along these lines have been intentionally omitted. Most likely, the further construction of linear spaces with high "dimension" will prove more useful in lowering the bound on $L(n, K_{n,n})$ than merely pushing the estimates of n versus q above. In fact, we conjecture that $L(n, K_{n,n})$ is actually universally bounded.

Conjecture 6. There exists a constant C such that $L(n, K_{n,n}) \leq C$ for all sufficiently large $n \geq 2$.

Perhaps even $C = 6$ is the truth, with $L(m, K_{n,n}) = 4$ or 6 for all meaningful m and n.

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